Differential Equation-Reduction of Order 可以降阶的高阶微分方程

High Order Derivatives

If the derivative f' of a function f is differentiable, then the derivative of f' is the second derivative of f represented by f'' (reads as f double prime). You can continue to differentiate f as long as there is differentiability.

$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	…
dy/dx	d^2y/dx^2	d^3y/dx^3	d^4y/dx^4	…
y'	y''	y'''	$y^{(4)}$	…
$D_x(y)$	$D_x^2(y)$	$D_x^3(y)$	$D_x^4(y)$	…

Note that $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dy'}{dx}$

Example 1: If $y=5x^3+7x-10$, find the first four derivatives.

$$\frac{dy}{dx}=15x^2+7$$

$$\frac{d^2y}{dx^2}=30x$$

$$\frac{d^3y}{dx^3}=30$$

$$\frac{d^4y}{dx^4}=0$$

$y^{(n)}=f(x)$型微分方程

$$y^{(n-1)}=\int f(x)+C_1$$

$$y^{(n-2)}=\int\left[\int f(x)+C_1\right]dx+C_2$$

Example 2: Find the general solution of $y'''=e^{2x}-\cos x$.

$$y''=\frac{1}{2}e^{2x}-\sin x+C$$

$$y'=\frac{1}{4}e^{2x}+\cos x+Cx+C_2$$

$$y=\frac{1}{8}e^{2x}+\sin x+C_1x^2+C_2x+C_3\quad\left(C_1=\frac{C}{2}\right)$$

Example 3: 质量为 m 的指点受力 F 的作用沿 Ox 轴做直线运动. 设力$F=F(t)$在开始时刻$t=0$时$F(0)=F_0$，随着时间 t 的增大，力 F 均匀的减小，指导$t=T$时，$F(T)=0$. 如果开始时质点位于原点，且初速度为 0，求指点的运动规律。

$$\begin{cases}t=0,F(0)=F_0\\t=T,F(T)=0\end{cases}\Rightarrow F(t)=kt+b\ \Rightarrow\begin{cases}b=F_0\\k=-F_0/T\end{cases}\Rightarrow\ F(t)=-\frac{F_0}{T}t+F_0$$

$$F(t)=ma=m\frac{d^2x}{dt^2}=-\frac{F_0}{T}t+F_0\ \Rightarrow\ \frac{d^2x}{dt^2}=\frac{F_0}{m}\left(1-\frac{t}{T}\right)$$

初始条件为

$$x|_{t=0}=0,\qquad v=\left.\frac{dx}{dt}\right|_{t=0}=0$$

积分可得

$$\frac{dx}{dt}=\int\frac{d^2x}{dt^2}dt=\int\left(\frac{F_0}{m}\left(1-\frac{t}{T}\right)\right)dt=\frac{F_0}{m}\left(t-\frac{t^2}{2T}\right)+C_1$$

带入 $v=\left.\frac{dx}{dt}\right|_{t=0}=0$的条件，可得

$$C_1=0\ \Rightarrow\ \frac{dx}{dt}=\frac{F_0}{m}\left(t-\frac{t^2}{2T}\right)\ \Rightarrow\ dx=\frac{F_0}{m}\left(t-\frac{t^2}{2T}\right)dt$$

积分可得

$$x=\int dx=\int\frac{F_0}{m}\left(t-\frac{t^2}{2T}\right)dt=\frac{F_0}{m}\left(\frac{t^2}{2}-\frac{t^3}{6T}\right)+C_2$$

带入 $x|_{t=0}=0$的条件，可得

$$C_2=0\ \Rightarrow\ x=\frac{F_0}{m}\left(\frac{t^2}{2}-\frac{t^3}{6T}\right)$$

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$y''=f(x,y')$型微分方程

此方程的特点是，方程的右端不显含未知数 y.

可以设$y'=p$

$$y''=\frac{dp}{dx}=p'$$

此时原方程可以化为

$$p'=f(x,p)$$

这是一个关于变量(x,p)的一阶微分方程。

Example 4: Find the solution of $(1+x^2)y''=2xy',\ y|_{x=0}=1,\ y'|_{x=0}=3$

Let $y'=p$, we have

$$(1+x^2)\frac{dp}{dx}=2xp$$

$$\frac{dp}{p}=\frac{2x}{1+x^2}$$

Integrate the function

$$\ln|p|=\ln(1+x^2)+C$$

$$p=y'=C_1(1+x^2)\quad (C_1=\pm e^c)$$

Plug in the initial condition $y'|_{x=0}=3$, we have

$$3=C_1(1+0^2)\ \Rightarrow\ C_1=3$$

$$y'=3(1+x^2)$$

Integrate the function

$$y=3\left(x+\frac{x^3}{3}\right)+C_2=x^3+3x+C_2$$

Plug in the initial condition $y|_{x=0}=1$, we have

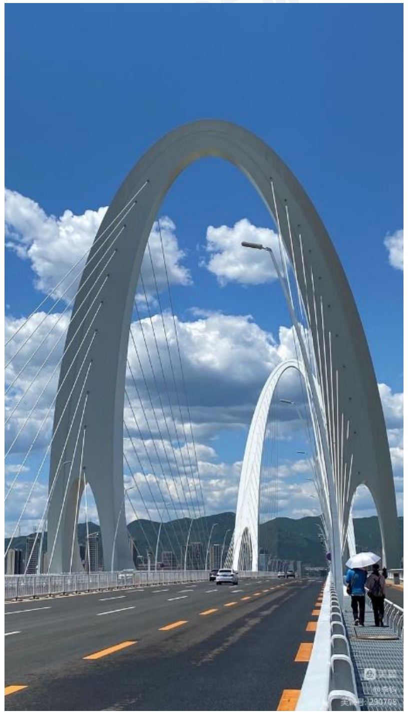
$$1=C_2$$

$$y=x^3+3x+1$$

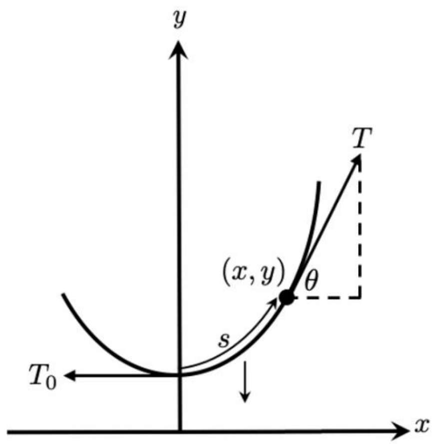
Example 5: Catenary 悬链线

In physics and geometry, a catenary (US: /ˈkætənɛri/, UK: /kəˈtiːnəri/) is the curve that an idealized hanging chain or cable assumes under its own weight when supported only at its ends in a uniform gravitational field.

The Shougang Bridge (Shijingshan District, Beijing) is a flattened catenary.



At the lowest position A, the tension only have the x-direction component T_0. The length of the string from point A to point (x,y) is s and its weight is $mg=\rho sg$. At point (x,y) the tension is T and its angle from x-direction is θ.



$$\begin{cases} x\ direction: T\cos\theta=T_0 \\ y\ direction: T\sin\theta=\rho gs \end{cases}$$

$$\Rightarrow\ \tan\theta=\rho gs/T_0$$

The curve length s is:

$$s=\int_0^x\sqrt{1+(y')^2}\,dx$$

$$\tan\theta=y'=\rho gs/T_0\Rightarrow\ s=(y'T_0)/\rho g$$

So we can have:

$$\int_0^x\sqrt{1+(y')^2}\,dx=\frac{y'T_0}{\rho g}$$

Taking the derivative for the equation above with respect of x, we can obtain:

$$\sqrt{1+(y')^2}=\frac{y''T_0}{\rho g}$$

上式是一个典型的$y''=f(x,y')$型微分方程，方程的右端不显含未知数 y.

Let $|OA|=a$, we can have the initial condition:

$$y|_{x=0}=a,\qquad y'|_{x=0}=0$$

Let $y'=p, y''=dp/dx$, we can obtain

$$\sqrt{1+(p)^2}=\frac{T_0}{\rho g}\left(\frac{dp}{dx}\right)$$

Separate variables

$$\frac{\rho g}{T_0}dx=\frac{dp}{\sqrt{1+p^2}}$$

Please notice that T_0 is constant, so we can integrate the differential equation:

$$\int\frac{\rho g}{T_0}dx=\int\frac{dp}{\sqrt{1+p^2}}$$

Integrate the function (Refer to P389, Calculus 10th ed, Ron Larson, of Inverse Hyperbolic Function Integration $\int\frac{du}{\sqrt{u^2+a^2}}=\ln(u+\sqrt{u^2+a^2})$ for details)

$$\ln(p+\sqrt{1+p^2})=\frac{\rho g}{T_0}x+C_1$$

Plug the initial condition into the equation above: $y'|_{x=0}=0\Rightarrow p|_{x=0}=0$

$$\ln(0+\sqrt{1+0^2})=C_1 \ \Rightarrow\ C_1=0$$

So we have

$$\ln(p+\sqrt{1+p^2})=\frac{\rho g}{T_0}x \ \Rightarrow\ p+\sqrt{1+p^2}=e^{(\frac{\rho g}{T_0}x)}$$

$$\Rightarrow\ \sqrt{1+p^2}=e^{(\frac{\rho g}{T_0}x)}-p\ \Rightarrow\ 1+p^2=e^{2(\frac{\rho g}{T_0}x)}-2pe^{(\frac{\rho g}{T_0}x)}+p^2$$

$$\Rightarrow\ 2pe^{(\frac{\rho g}{T_0}x)}=e^{2(\frac{\rho g}{T_0}x)}-1\ \Rightarrow p=\frac{1}{2}\left(e^{(\frac{\rho g}{T_0}x)}-e^{-(\frac{\rho g}{T_0}x)}\right)=\sinh(\frac{\rho g}{T_0}x)$$

The notation $\sinh x$ is read as "the hyperbolic sine of x"

Integrate the equation and we can obtain:

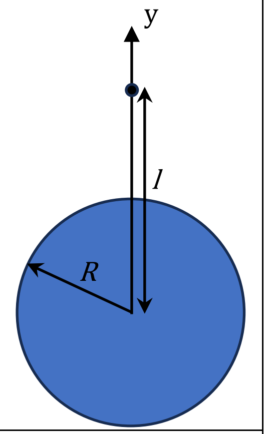
$$y=\frac{T_0}{2\rho g}\left(e^{(\frac{\rho g}{T_0}x)}+e^{-(\frac{\rho g}{T_0}x)}\right)+C_2=\frac{T_0}{2\rho g}\cosh(\frac{\rho g}{T_0}x)+C_2$$

Plug the initial condition into the equation above: $y|_{x=0}=a$

$$C_2=0$$

$$y=\frac{T_0}{2\rho g}\left(e^{(\frac{\rho g}{T_0}x)}+e^{-(\frac{\rho g}{T_0}x)}\right)=\frac{T_0}{2\rho g}\cosh(\frac{\rho g}{T_0}x)$$

Exercise 1: 某距离地面很高的物体，受到地心引力的作用，从静止开始坠向地球。忽略空气阻力，设地球半径是 R,物体开始下坠时距离地心为 l　(l>R)，地球质量为 M，地面附件的重力加速度为 g，求此物体降落到地面所需要的时间。



设物体质量 m，根据万有引力定律，物体所受到的引力为:

$$F=-\frac{GMm}{y^2}$$

根据牛顿第二定律，此物体的加速度为

$$F=ma=mv'=my''$$

Initial condition:

$$\begin{cases}\text{初始位置为 l：}\ y|_{t=0}=l\\ \text{初始速度为 0：}\ y'|_{t=0}=0\\ \text{地表加速度为}g:\ y''|_{y=R}=g\end{cases}$$

列出微分方程

$$-\frac{GMm}{y^2}=my''\ \Rightarrow\ -\frac{GM}{y^2}=y''$$